UR-1445
US-FT-28/95
hep-th/9511091

arXiv:hep-th/9511091v1 13 Nov 1995

The Zero Curvature Formulation of TB, sTB Hierarchy and Topological Algebras

Ashok Das

Department of Physics and Astronomy
University of Rochester
Rochester, N.Y. 14627, USA

and

Shibaji Roy *

Departamento de Física de Partículas
Universidade de Santiago
E-15706 Santiago de Compostela, Spain

ABSTRACT

A particular dispersive generalization of long water wave equation in $1+1$ dimensions, which is important in the study of matrix models without scaling limit, known as two–Boson (TB) equation, as well as the associated hierarchy has been derived from the zero curvature condition on the gauge group $SL(2,R)\otimes U(1)$. The supersymmetric extension of the two–Boson (sTB) hierarchy has similarly been derived from the zero curvature condition associated with the gauge supergroup $OSp(2|2)$. Topological algebras arise naturally as the second Hamiltonian structure of these classical integrable systems, indicating a close relationship of these models with 2d topological field theories.

*E-mail address: roy@gaes.usc.es

Integrable models [1–4] in $1+1$ dimensions play very interesting and sometimes mysterious roles in the study of conformal field theories [5], matrix model formulation of string theories [6], 2d topological field theories [7,8] and the intersection theory on the moduli space of Riemann surfaces [9,10]. These models contain very rich mathematical structures and deserve to be studied on their own right. Recently, a particular dispersive generalization of long water wave equation in narrow channel, known as the TB equation [11] has attracted some attention due to its relation with the matrix models without the continuum limit [12]. This relation has been interpreted as an indication of the topological nature of the latter models.

The TB–hierarchy as well as their close relatives have been studied quite extensively in the literature [11,13,14]. In particular, TB–hierarchy was constructed in ref.[11] in a non–standard Lax operator approach and has been shown to possess a tri–Hamiltonian structure. The supersymmetric extension of TB–hierarchy or sTB–hierarchy and many of their interesting properties have also been studied recently [15,16] in the non–standard superLax operator approach. Integrable models, on the otherhand, are also known [17] to be obtained from the group theoretical point of view where the dynamical equations are obtained from the zero curvature condition associated with some symmetry group. While some properties of an integrable model can be understood in the Lax operator approach, the bi–Hamiltonian structure, integrabilty property and the geometry of the phase–space become more transparent in the zero curvature formulation [18]. Moreover, ordinary 2d gravity theories and their underlying current algebra structure in the light cone gauge can be better understood by connecting them to integrable models in the zero curvature approach [19,20]. Thus, these two approaches play complementary roles with each other.

In this paper we study both TB and sTB–hierarchy in the zero curvature approach. We derive TB–hierarchy from the zero curvature condition associated with the gauge group $SL(2,R) \otimes U(1)$ and similarly sTB–hierarchy is obtained from the gauge supergroup $OSp(2|2)$. We point out that both these integrable systems possess two bi–Hamiltonian structures. The second Hamiltonian structure of the TB–hierarchy is shown to be $U(1)$ extended Virasoro algebra with zero central charge and that of sTB–hierarchy is the twisted $N=2$ superconformal algebra. This observation, therefore, indicates that these integrable models are closely related to 2d topological field theories.

The dispersive long water wave equation (TB equation) with which we will be con-

cerned in this paper has the form:

$$\begin{aligned}
\frac{\partial J_1}{\partial t} &= (J_1' - 2J_1 J_0)' \\
\frac{\partial J_0}{\partial t} &= \left(2J_1 - J_0' - J_0^2\right)'
\end{aligned} \tag{1}$$

These are coupled differential equations where $J_1(x,t)$ and $J_0(x,t)$ refer respectively the horizontal velocity and the height of the free water surface and a prime denotes derivative with respect to x. By choosing the scaling dimension of x to be -1, we find from eq.(1) that the scaling dimensions of t, $J_1(x,t)$ and $J_0(x,t)$ are fixed as -2, 2 and 1 respectively. This observation will be helpful in choosing the gauge fixing condition in our zero curvature analysis.

Now we show how the TB equation (1) as well as the hierarchy associated with it can be obtained from the zero curvature condition on a special gauge fixed form of $SL(2,R) \otimes U(1)$ Lie algebra valued gauge fields. There are four bosonic generators denoted by t_a in $SL(2,R) \otimes U(1)$ Lie algebra, where $a = 0, \pm$ are the $SL(2,R)$ indices and $a = (0)$ is the $U(1)$ index. The algebra can be written as,

$$[t_a,\, t_b] \;=\; f_{ab}^{c}\, t_c \tag{2}$$

where the non-zero structure constants are $f_{+0}^{+} = f_{0-}^{-} = \frac{1}{2} f_{+-}^{0} = 1$. The zero curvature condition in $1+1$ dimensions associated with the Lie algebra valued gauge fields $A_\mu \equiv A_\mu^a t_a$, where $\mu = t,\, x$ denote respectively the time and space components, has the form:

$$F_{tx} \;=\; \partial_t A_x - \partial_x A_t + [A_t,\, A_x] \;=\; 0. \tag{3}$$

Written in components we get four equations from (3) as given below,

$$\partial_t A_x^+ - \partial_x A_t^+ - A_t^0 A_x^+ + A_t^+ A_x^0 \;=\; 0 \tag{4}$$

$$\partial_t A_x^- - \partial_x A_t^- + A_t^0 A_x^- - A_t^- A_x^0 \;=\; 0 \tag{5}$$

$$\partial_t A_x^0 - \partial_x A_t^0 + 2A_t^+ A_x^- - 2A_t^- A_x^+ \;=\; 0 \tag{6}$$

$$\partial_t A_x^{(0)} - \partial_x A_t^{(0)} \;=\; 0 \tag{7}$$

Identifying $A_x^+ \equiv J_1(x,t)$ and $A_x^{(0)} \equiv J_0(x,t)$ as the two dynamical variables of the TB equation (1), and noting that they have scaling dimensions 2 and 1 respectively as mentioned before, we get from (4)–(7) the scaling dimensions of the other two components

of the gauge field as $[A_x^-] = 0$ and $[A_x^0] = 1$. We, therefore, choose the gauge fixing conditions as,

$$A_x^+ \;=\; J_1(x,t), \quad A_x^- \;=\; -1, \quad A_x^0 \;=\; J_0(x,t), \quad A_x^{(0)} \;=\; J_0(x,t) \tag{8}$$

With these gauge fixing conditions, we note that from eq.(5) and from a combination of (6) and (7) we get two constraint equations which can be solved as,

$$\begin{aligned} A_t^0 &= -\partial_x A_t^- - J_0 A_t^- \\ A_t^+ &= \frac{1}{2}\partial_x A_t^{(0)} + \frac{1}{2}\partial_x^2 A_t^- + \frac{1}{2}J_0' A_t^- + \frac{1}{2}J_0\partial_x A_t^- - J_1 A_t^- \end{aligned} \tag{9}$$

Substituting these in (4) and (7), we obtain two dynamical equations as follows,

$$\begin{aligned} \partial_t J_1 &= \frac{1}{2}\partial_x^2 A_t^{(0)} + \frac{1}{2}\partial_x^3 A_t^- + \frac{1}{2}J_0'' A_t^- + J_0'\partial_x A_t^- - J_1' A_t^- \\ &\quad -2J_1\partial_x A_t^- - \frac{1}{2}J_0\partial_x A_t^{(0)} - \frac{1}{2}J_0 J_0' A_t^- - \frac{1}{2}J_0^2\partial_x A_t^- \\ \partial_t J_0 &= \partial_x A_0^{(0)} \end{aligned} \tag{10}$$

In (10) we have obtained the dynamical equations in terms of two independent functions $A_t^-(J_1, J_0)$ and $A_t^{(0)}(J_1, J_0)$. We now make a redefinition of the function $A_t^{(0)}$ in terms of the old variables $A_t^{(0)}$ and A_t^- as given below,

$$\tilde{A}_t^{(0)} \;=\; \frac{1}{2}A_t^{(0)} + \frac{1}{2}\partial_x A_t^- + \frac{1}{2}J_0 A_t^- \tag{11}$$

With this redefinition the dynamical equations (10) reduce to the following form:

$$\partial_t J_1 \;=\; \partial_x^2 A_t^{(0)} - J_0\partial_x A_t^{(0)} - J_1' A_t^- - 2J_1\partial_x A_t^- \tag{12}$$

$$\partial_t J_0 \;=\; 2\partial_x A_t^{(0)} - \partial^2 A_t^- - J_0' A_t^- - J_0\partial_x A_t^- \tag{13}$$

where in writing (12) and (13), we have omitted 'tilde' from $A_t^{(0)}$. These equations define the TB–hierarchy. We now show how to extract the explicit forms of the equations of the hierarchy as well as the two bi-Hamiltonian structures mentioned earlier from these hierarchy equations.

By shifting $J_0(x,t) \to J_0(x,t) + \lambda$, where λ is a space–time independent parameter of scaling dimension 1, known as the spectral parameter, we obtain from the hierarchy equations (12) and (13),

$$\begin{aligned} \partial_t J_1 &= \partial_x^2 A_t^{(0)} - J_0\partial_x A_t^{(0)} - J_1' A_t^- - 2J_1\partial_x A_t^- - \lambda\partial_x A_t^{(0)} \\ \partial_t J_0 &= 2\partial_x A_t^{(0)} - \partial^2 A_t^- - J_0' A_t^- - J_0\partial_x A_t^- - \lambda\partial_x A_t^- \end{aligned} \tag{14}$$

Since the dynamical variables $J_1(x,t)$ and $J_0(x,t)$ are independent of λ, we obtain a set of recursion relation and dynamical equation from (14) if we expand

$$\begin{aligned} A_t^-(J_1, J_0, \lambda) &= \sum_{j=0}^{n} A_j(J_1, J_0)\lambda^{n-j} \\ A_t^{(0)}(J_1, J_0, \lambda) &= \sum_{j=0}^{n} B_j(J_1, J_0)\lambda^{n-j} \end{aligned}$$

as follows,

$$\begin{aligned} -\left(J_1' + 2J_1\partial_x\right) A_j + \left(\partial_x^2 - J_0\partial_x\right) B_j &= \partial_x B_{j+1} \\ -\left(\partial_x^2 + J_0' + J_0\partial_x\right) A_j + 2\partial_x B_j &= \partial_x A_{j+1} \qquad \text{for } j = 0, 1, \ldots, n-1 \end{aligned} \tag{15}$$

and for $j = n$

$$\begin{aligned} \partial_t J_1 &= -\left(J_1' + 2J_1\partial_x\right) A_n + \left(\partial_x^2 - J_0\partial_x\right) B_n \\ \partial_t J_0 &= -\left(\partial_x^2 + J_0' + J_0\partial_x\right) A_n + 2\partial_x B_n \end{aligned} \tag{16}$$

From these two sets of equations (15) and (16), we can generate the whole hierarchy associated with TB equation as follows. It is clear from (15) that A_0 and B_0 are both constants and therefore we choose $A_0 = B_0 = -1$. Substituting these values in (16) we obtain the zeroth order dynamical equation as given below,

$$\begin{aligned} \partial_t J_1 &= J_1' \\ \partial_t J_0 &= J_0' \end{aligned} \tag{17}$$

which are nothing but the chiral wave equations. Then using the values of A_0 and B_0 in the recursion relations (15) we obtain $A_1 = J_0$ and $B_1 = J_1$. Substituting these values in (16) we get the first order dynamical equations as,

$$\begin{aligned} \partial_t J_1 &= \left(J_1' - 2J_1 J_0\right)' \\ \partial_t J_0 &= \left(2J_1 - J_0' - J_0^2\right)' \end{aligned} \tag{18}$$

These are precisely the TB equation written earlier in (1). This procedure, therefore, generates the TB–hierarchy. By identifying $A_n \equiv \frac{\delta H_n}{\delta J_1}$ and $B_n \equiv \frac{\delta H_n}{\delta J_0}$, where δ indicates the variational derivatives and H_n's are the Hamiltonians (conserved quantities) associated with the integrable system, we can easily recover from (15) and (16) the two Hamiltonian

structures (Poisson brackets) among the dynamical variables as,

$$\begin{aligned}\{J_1(x),\, J_0(y)\}_1 &= \partial_x \delta(x-y) \\ \{J_1(x),\, J_1(y)\}_1 &= \{J_0(x),\, J_0(y)\}_1 \;=\; 0\end{aligned} \tag{19}$$

and

$$\begin{aligned}\{J_1(x),\, J_1(y)\}_2 &= -\left[J_1'(x) + 2J_1(x)\partial_x\right]\delta(x-y) \\ \{J_1(x),\, J_0(y)\}_2 &= \left[\partial_x^2 - J_0(x)\partial_x\right]\delta(x-y) \\ \{J_0(x),\, J_0(y)\}_2 &= 2\partial_x \delta(x-y)\end{aligned} \tag{20}$$

Using the Poisson bracket structures, one can easily work out the explicit forms of the Hamiltonians H_n's, in order to write the dynamical equations as Hamilton's equation of motion. This way one can regard the integrable systems as Hamiltonian systems. In order to extract the second bi-Hamiltonian system, we can proceed exactly as in the previous case except that we now shift instead, the other dynamical variable $J_1(x,t)$ by the spectral parameter as, $J_1(x,t) \to J_1(x,t) + \lambda^2$. Now expanding the independent functions in terms of the spectral paramter as

$$\begin{aligned}A_t^-(J_1, J_0, \lambda^2) &= \sum_{j=0}^{n} A_j(J_1, J_0)(\lambda^2)^{n-j} \\ A_t^{(0)}(J_1, J_0, \lambda^2) &= \sum_{j=0}^{n} B_j(J_1, J_0)(\lambda^2)^{n-j}\end{aligned}$$

we obtain the recursion relation in the following form

$$\begin{aligned}-\left(J_1' + 2J_1\partial_x\right)A_j + \left(\partial_x^2 - J_0\partial_x\right)B_j &= 2\partial_x A_{j+1} \\ -\left(\partial_x^2 + J_0' + J_0\partial_x\right)A_j + 2\partial_x B_j &= 0 \qquad \text{for } j = 0, 1, \ldots, n-1\end{aligned} \tag{21}$$

and for $j = n$, the dynamical equations remain the same as eq.(16). It is clear from (21) that in this case A_0 is a constant and B_n's get fixed from the second equation of (21). Choosing $A_0 = -1$, we get the zeroth and first order equations as,

$$\begin{aligned}\partial_t J_1 &= \left(-\frac{1}{2}J_0' + \frac{1}{4}J_0^2 + J_1\right)' \\ \partial_t J_0 &= 0\end{aligned} \tag{22}$$

and

$$\begin{aligned}\partial_t J_1 &= -\frac{1}{8}J_0'''' + \frac{3}{4}J_0(J_0')^2 + \frac{3}{8}J_0^2 J_0'' + J_1 J_0'' + \frac{5}{4}J_1' J_0' + \frac{1}{8}J_0 J_0''' \\ &\quad -\frac{3}{8}J_0^3 J_0' + J_1 J_0 J_0' - \frac{5}{8}J_1' J_0^2 - \frac{3}{2}J_1 J_1' + \frac{1}{2}J_1''' \\ \partial_t J_0 &= 0\end{aligned} \tag{23}$$

We note that in this case one of the dynamical variables $J_0(x,t)$ does not evolve with time. So, setting it to zero, we recover from (23) the KdV equation. This shows how KdV–hierarchy is contained in TB–hierarchy. The first Hamiltonian structure can be seen from (21) to have the form:

$$\begin{aligned}\{J_1(x),\, J_1(y)\}_1 &= 2\partial_x\delta(x-y)\\ \{J_1(x),\, J_0(y)\}_1 &= \{J_0(x),\, J_0(y)\}_1 \;=\; 0\end{aligned}\tag{24}$$

and the second Hamiltonian structure remains as given before in (20). This analysis, therefore, brings out the two bi–Hamiltonian structures associated with the TB–hierarchy. We can recognize the topological nature of the second Hamiltonian structure by observing that (20) represents a $U(1)$ extended Virasoro algebra with zero central charge where the $U(1)$ current is anomalous. This topological symmetry will become more manifest when we go to the supersymmetric case i.e. sTB–hierarchy which we consider next.

We will show that sTB–hierarchy can be generated from the zero curvature condition on the gauge supergroup $OSp(2|2)$ which consists of four bosonic and four fermionic generators. The corresponding Lie superalgebra is given as

$$[t_i,\, t_j] \;=\; f_{ij}^{k}t_k; \quad [t_i,\, t_\alpha] \;=\; f_{i\alpha}^{\beta}t_\beta; \quad [t_\alpha,\, t_\beta]_+ \;=\; f_{\alpha\beta}^{i}t_i; \tag{25}$$

where the subscript '+' denotes the anticommutator. The bosonic indices i, j take values 0, $\pm$, (0) and the fermionic indices α, β take $\pm\frac{1}{2}$, $(\pm\frac{1}{2})$. The non–zero structure constants are $f_{+0}^{+} = f_{0-}^{-} = f_{(0)\frac{1}{2}}^{\frac{1}{2}} = f_{(0)-\frac{1}{2}}^{-\frac{1}{2}} = f_{\frac{1}{2}(0)}^{(\frac{1}{2})} = f_{(-\frac{1}{2})(0)}^{(-\frac{1}{2})} = f_{\frac{1}{2}(-\frac{1}{2})}^{0} = f_{-\frac{1}{2}(\frac{1}{2})}^{0} = f_{-\frac{1}{2}(-\frac{1}{2})}^{-} = f_{\frac{1}{2}(\frac{1}{2})}^{+} = f_{\frac{1}{2}-}^{-\frac{1}{2}} = f_{+-\frac{1}{2}}^{\frac{1}{2}} = f_{+(-\frac{1}{2})}^{(\frac{1}{2})} = f_{\frac{1}{2}-}^{(-\frac{1}{2})} = 2f_{\frac{1}{2}(-\frac{1}{2})}^{(0)} = 2f_{\frac{1}{2}0}^{\frac{1}{2}} = 2f_{0-\frac{1}{2}}^{-\frac{1}{2}} = 2f_{(\frac{1}{2})0}^{(\frac{1}{2})} = 2f_{0(-\frac{1}{2})}^{(-\frac{1}{2})} = -2f_{(\frac{1}{2})-\frac{1}{2}}^{0} = \frac{1}{2}f_{+-}^{0} = 1$. So, the zero curvature condition (3) in this case in components takes the form:

$$\partial_t A_x^+ - \partial_x A_t^+ + A_t^+ A_x^0 - A_t^0 A_x^+ + A_t^{\frac{1}{2}} A_x^{(\frac{1}{2})} + A_t^{(\frac{1}{2})} A_x^{\frac{1}{2}} \;=\; 0 \tag{26}$$

$$\partial_t A_x^- - \partial_x A_t^- + A_t^0 A_x^- - A_t^- A_x^0 + A_t^{-\frac{1}{2}} A_x^{(-\frac{1}{2})} + A_t^{(-\frac{1}{2})} A_x^{-\frac{1}{2}} \;=\; 0 \tag{27}$$

$$\begin{aligned}\partial_t A_x^0 - \partial_x A_t^0 + A_t^{\frac{1}{2}} A_x^{(-\frac{1}{2})} + A_t^{(-\frac{1}{2})} A_x^{\frac{1}{2}} + A_t^{-\frac{1}{2}} A_x^{(\frac{1}{2})} + A_t^{(\frac{1}{2})} A_x^{-\frac{1}{2}}&\\ +2A_t^+ A_x^- - 2A_t^- A_x^+ &\;=\; 0\end{aligned} \tag{28}$$

$$\partial_t A_x^{(0)} - \partial_x A_t^{(0)} - \frac{1}{2}A_t^{(\frac{1}{2})} A_x^{-\frac{1}{2}} - \frac{1}{2}A_t^{-\frac{1}{2}} A_x^{(\frac{1}{2})} + \frac{1}{2}A_t^{\frac{1}{2}} A_x^{(-\frac{1}{2})} + \frac{1}{2}A_t^{(-\frac{1}{2})} A_x^{\frac{1}{2}} \;=\; 0 \tag{29}$$

$$\begin{aligned}\partial_t A_x^{\frac{1}{2}} - \partial_x A_t^{\frac{1}{2}} + A_t^{(0)} A_x^{\frac{1}{2}} - A_t^{\frac{1}{2}} A_x^{(0)} + A_t^+ A_x^{-\frac{1}{2}} - A_t^{-\frac{1}{2}} A_x^+&\\ +\frac{1}{2}A_t^{\frac{1}{2}} A_x^0 - \frac{1}{2}A_t^0 A_x^{\frac{1}{2}} &\;=\; 0\end{aligned} \tag{30}$$

$$
\begin{aligned}
\partial_t A_x^{-\frac{1}{2}} - \partial_x A_t^{-\frac{1}{2}} + A_t^{\frac{1}{2}} A_x^- - A_t^- A_x^{\frac{1}{2}} + A_t^{(0)} A_x^{-\frac{1}{2}} - A_t^{-\frac{1}{2}} A_x^{(0)} & \\
+\frac{1}{2} A_t^0 A_x^{-\frac{1}{2}} - \frac{1}{2} A_t^{-\frac{1}{2}} A_x^0 &= 0 \qquad (31)\\
\partial_t A_x^{(\frac{1}{2})} - \partial_x A_t^{(\frac{1}{2})} + A_t^+ A_x^{(-\frac{1}{2})} - A_t^{(-\frac{1}{2})} A_x^+ + A_t^{(\frac{1}{2})} A_x^{(0)} - A_t^{(0)} A_x^{(\frac{1}{2})} & \\
+\frac{1}{2} A_t^{(\frac{1}{2})} A_x^0 - \frac{1}{2} A_t^0 A_x^{(\frac{1}{2})} &= 0 \qquad (32)\\
\partial_t A_x^{(-\frac{1}{2})} - \partial_x A_t^{(-\frac{1}{2})} + A_t^{(\frac{1}{2})} A_x^- - A_t^- A_x^{(\frac{1}{2})} + A_t^{(-\frac{1}{2})} A_x^{(0)} - A_t^{(0)} A_x^{(-\frac{1}{2})} & \\
+\frac{1}{2} A_t^0 A_x^{(-\frac{1}{2})} - \frac{1}{2} A_t^{(-\frac{1}{2})} A_x^0 &= 0 \qquad (33)
\end{aligned}
$$

Identifying $A_x^+ \equiv J_1(x,t)$, $A_x^{(0)} \equiv \frac{1}{2} J_0(x,t)$ as the bosonic variables of sTB–hierarchy and $A_x^{(\frac{1}{2})} \equiv \xi(x,t)$, $A_x^{\frac{1}{2}} \equiv \bar{\xi}(x,t)$ as the fermionic variables, a simple dimensional analysis from (26)–(33) allows us to choose the following gauge fixing conditions for the rest of the gauge field components,

$$
A_x^- = -1, \quad A_x^0 = J_0(x,t), \quad A_x^{-\frac{1}{2}} = A_x^{(-\frac{1}{2})} = 0 \qquad (34)
$$

Substituting these values, we find that eqs.(27), (31), (33) and a combination of (28) and (29) give four constraints which can be solved to obtain,

$$
\begin{aligned}
A_t^0 &= -\partial_x A_t^- - A_t^- J_0 \\
A_t^{\frac{1}{2}} &= -\partial_x A_t^{-\frac{1}{2}} - A_t^- \bar{\xi} - A_t^{-\frac{1}{2}} J_0 \\
A_t^{(\frac{1}{2})} &= -\partial_x A_t^{(-\frac{1}{2})} - A_t^- \xi \qquad (35)\\
A_t^+ &= \frac{1}{2}\partial_x A_t^{(0)} + \frac{1}{2}\partial_x^2 A_x^- + A_t^{-\frac{1}{2}} \xi + \frac{1}{2}\partial_x A_t^- J_0 + \frac{1}{2} A_t^- J_0' - A_t^- J_1
\end{aligned}
$$

and the other equations (26), (29), (30) and (32) give the dynamical equations in the following form

$$
\begin{aligned}
\partial_t J_1 &= -\left(J_1' + 2J_1\partial_x\right) A_t^- + \left(\partial_x^2 - J_0\partial_x\right) A_t^{(0)} - \left(\xi' + 2\xi\partial_x\right) A_t^{-\frac{1}{2}} - \bar{\xi}\partial_x A_t^{(-\frac{1}{2})} \\
\partial_t J_0 &= -\left(\partial_x^2 + J_0\partial_x + J_0'\right) A_t^- + 2\partial_x A_t^{(0)} - \xi A_t^{-\frac{1}{2}} + \bar{\xi} A_t^{(-\frac{1}{2})} \\
\partial_t \xi &= -\left(2\xi\partial_x + \xi'\right) A_t^- + \xi A_t^{(0)} - \left(\partial_x^2 - J_1 - J_0\partial_x\right) A_t^{(-\frac{1}{2})} \qquad (36)\\
\partial_t \bar{\xi} &= -\left(\bar{\xi}\partial_x + \bar{\xi}'\right) A_t^- - \bar{\xi} A_t^{(0)} - \left(\partial_x^2 + J_0\partial_x + J_0' - J_1\right) A_t^{-\frac{1}{2}}
\end{aligned}
$$

In obtaining (36) we have made use of (35) and the same field redefinition (11) as in the bosonic case. Note that we have obtained the dynamical equations in terms of four independent functions A_t^-, $A_t^{(0)}$, $A_t^{-\frac{1}{2}}$ and $A_t^{(-\frac{1}{2})}$ and these equations define the sTB–hierarchy. In order to extract the explicit forms of the equations we use the same trick as

in the bosonic case. The first bi–Hamiltonian structure and the first set of equations can be obtained by shifting $J_0(x,t) \to J_0(x,t) + \lambda$. Again expanding the four independent functions in powers of λ as,

$$
\begin{aligned}
A_t^{-}(J_1, J_0, \xi, \bar{\xi}, \lambda) &= \sum_{j=0}^{n} A_j(J_1, J_0, \xi, \bar{\xi})\lambda^{n-j} \\
A_t^{(0)}(J_1, J_0, \xi, \bar{\xi}, \lambda) &= \sum_{j=0}^{n} B_j(J_1, J_0, \xi, \bar{\xi})\lambda^{n-j} \\
A_t^{-\frac{1}{2}}(J_1, J_0, \xi, \bar{\xi}, \lambda) &= \sum_{j=0}^{n} \alpha_j(J_1, J_0, \xi, \bar{\xi})\lambda^{n-j} \\
A_t^{(-\frac{1}{2})}(J_1, J_0, \xi, \bar{\xi}, \lambda) &= \sum_{j=0}^{n} \beta_j(J_1, J_0, \xi, \bar{\xi})\lambda^{n-j}
\end{aligned}
$$

we obtain the following recursion relations for $j = 0, 1, \ldots, n-1$

$$
\begin{aligned}
-\left(J_1' + 2J_1\partial_x\right) A_j + \left(\partial_x^2 - J_0\partial_x\right) B_j - \left(\xi' + 2\xi\partial_x\right)\alpha_j - \bar{\xi}\partial_x\beta_j &= \partial_x B_{j+1} \\
-\left(\partial_x^2 + J_0\partial_x + J_0'\right) A_j + 2\partial_x B_j - \xi\alpha_j + \bar{\xi}\beta_j &= \partial_x A_{j+1} \\
-\left(2\xi\partial_x + \xi'\right) A_j + \xi B_j - \left(\partial_x^2 - J_1 - J_0\partial_x\right)\beta_j &= -\partial_x\beta_{j+1} \\
-\left(\bar{\xi}\partial_x + \bar{\xi}'\right) A_j - \bar{\xi}B_j - \left(\partial_x^2 + J_0\partial_x + J_0' - J_1\right)\alpha_j &= \partial_x\alpha_{j+1}
\end{aligned}
\qquad (37)
$$

and the following dynamical equations for $j = n$,

$$
\begin{aligned}
\partial_t J_1 &= -\left(J_1' + 2J_1\partial_x\right) A_n + \left(\partial_x^2 - J_0\partial_x\right) B_n - \left(\xi' + 2\xi\partial_x\right)\alpha_n - \bar{\xi}\partial_x\beta_n \\
\partial_t J_0 &= -\left(\partial_x^2 + J_0\partial_x + J_0'\right) A_n + 2\partial_x B_n - \xi\alpha_n + \bar{\xi}\beta_n \\
\partial_t \xi &= -\left(2\xi\partial_x + \xi'\right) A_n + \xi B_n - \left(\partial_x^2 - J_1 - J_0\partial_x\right)\beta_n \\
\partial_t \bar{\xi} &= -\left(\bar{\xi}\partial_x + \bar{\xi}'\right) A_n - \bar{\xi}B_n - \left(\partial_x^2 + J_0\partial_x + J_0' - J_1\right)\alpha_n
\end{aligned}
\qquad (38)
$$

From (37) it is clear that A_0, B_0, α_0 and β_0 are constants independent of x. We, therefore, choose $A_0 = -1$, $B_0 = \alpha_0 = \beta_0 = 0$ and obtain from (38) the zeroth order equations as,

$$
\begin{aligned}
\partial_t J_1 &= J_1' \\
\partial_t J_0 &= J_0' \\
\partial_t \xi &= \xi' \\
\partial_t \bar{\xi} &= \bar{\xi}'
\end{aligned}
\qquad (39)
$$

This is the super extension of chiral wave equation obtained in (17). Substituting the above zero mode values in the recursion relation (37) we obtain $A_1 = J_0(x,t)$, $B_1 =$

$J_1(x,t)$, $\alpha_1 = \bar{\xi}(x,t)$ and $\beta_1 = -\xi(x,t)$, So, the first order dynamical equations as calculated from (38) have the form

$$
\begin{aligned}
\partial_t J_1 &= \left(J_1' - 2J_1J_0 - 2\xi\bar{\xi}\right)' \\
\partial_t J_0 &= \left(2J_1 - J_0' - J_0^2\right)' \\
\partial_t \xi &= \left(\xi' - 2J_0\xi\right)' \\
\partial_t \bar{\xi} &= \left(-\bar{\xi}' - 2J_0\bar{\xi}\right)'
\end{aligned}
\tag{40}
$$

We recognize these equations as the sTB equation [15]. Thus the whole hierarchy associated with the sTB equation can be generated from the recursion relations (37) and the dynamical equations (38). We would also like to point out here that the sTB fermions ξ and $\bar{\xi}$ do not have specific scaling dimensions as can be seen from (40). From the first equation of (40) we note that their sum would have to be 3. So, they could be either $\frac{3}{2}$ each like $N=2$ sKdV fermions or they could be 2 and 1 like the fermionic fields of topological field theories. It should be mentioned here that sTB equation (40) can be reduced to one of the lower equations of $N=2$ sKdV–hierarchy (eq.(2.33) of ref.[20]) by the following redefinition and rescaling of fields, $\partial_t \to -i\partial_t$, $\partial_x \to \partial_x$, $J_1 \to (u + i\phi')$, $J_0 \to 2i\phi$, $\xi \to (\xi_1 + i\xi_2)$ and $\bar{\xi} \to (\xi_1 - i\xi_2)$. The next equation of the sTB–hierarchy with the same redefinitions reduce to $a=4$, $N=2$ sKdV equation [20,21]. Thus sTB–hierarchy can be thought of as a twisted $N=2$ sKdV–hierarchy.

Now identifying $A_n \equiv \frac{\delta H_n}{\delta J_1}$, $B_n \equiv \frac{\delta H_n}{\delta J_0}$, $\alpha_n \equiv \frac{\delta H_n}{\delta \xi}$ and $\beta_n \equiv \frac{\delta H_n}{\delta \bar{\xi}}$, the two Hamiltonian structures can be easily read out from (37) to be

$$
\begin{aligned}
\{J_1(x),\, J_0(y)\}_1 &= \partial_x \delta(x-y) \\
\{\xi(x),\, \bar{\xi}(y)\}_1 &= -\partial_x \delta(x-y)
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
\{J_1(x),\, J_1(y)\}_2 &= -\left[J_1'(x) + 2J_1(x)\partial_x\right]\delta(x-y) \\
\{J_1(x),\, J_0(y)\}_2 &= \left[\partial_x^2 - J_0(x)\partial_x\right]\delta(x-y) \\
\{J_1(x),\, \xi(y)\}_2 &= -\left[\xi'(x) + 2\xi(x)\partial_x\right]\delta(x-y) \\
\{J_1(x),\, \bar{\xi}(y)\}_2 &= -\bar{\xi}(x)\partial_x\delta(x-y) \\
\{J_0(x),\, \xi(y)\}_2 &= -\xi(x)\delta(x-y) \\
\{J_0(x),\, \bar{\xi}(y)\}_2 &= \bar{\xi}(x)\delta(x-y)
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\{J_0(x),\, J_0(y)\}_2 &= 2\partial_x\delta(x-y) \\
\{\xi(x),\, \bar{\xi}(y)\}_2 &= -\left[\partial_x^2 - J_0(x)\partial_x - J_1(x)\right]\delta(x-y)
\end{aligned}
$$

The second Hamiltonian structure (42) is nothing but the twisted $N=2$ superconformal algebra or topological algebra where ξ and $\bar{\xi}$ have conformal dimensions 2 and 1 respectively. We would like to comment that the twisted $N=2$ superconformal algebra [8] contains a finite subalgebra consisting of four bosonic and four fermionic generators much like $OSp(2|2)$ algebra but is not isomorphic to it and may be called 'twisted' $OSp(2|2)$ algebra. It would have been more natural to expect the sTB–hierarchy to originate from the zero curvature condition on 'twisted' $OSp(2|2)$ rather than $OSp(2|2)$ itself. We have not been able to find any suitable gauge fixing which will generate sTB–hierarchy this way.

As in the bosonic case the second bi–Hamiltonian structure can be obtained by shifting $J_1(x,t) \to J_1(x,t) + \lambda^2$. Expanding the functions as,

$$
\begin{aligned}
A_t^-(J_1, J_0, \xi, \bar{\xi}, \lambda^2) &= \sum_{j=0}^{n} A_j(J_1, J_0, \xi, \bar{\xi})(\lambda^2)^{n-j} \\
A_t^{(0)}(J_1, J_0, \xi, \bar{\xi}, \lambda^2) &= \sum_{j=0}^{n} B_j(J_1, J_0, \xi, \bar{\xi})(\lambda^2)^{n-j} \\
A_t^{-\frac{1}{2}}(J_1, J_0, \xi, \bar{\xi}, \lambda^2) &= \sum_{j=0}^{n} \alpha_j(J_1, J_0, \xi, \bar{\xi})(\lambda^2)^{n-j} \\
A_t^{(-\frac{1}{2})}(J_1, J_0, \xi, \bar{\xi}, \lambda^2) &= \sum_{j=0}^{n} \beta_j(J_1, J_0, \xi, \bar{\xi})(\lambda^2)^{n-j}
\end{aligned}
$$

we obtain the following recursion relations from the hierarchy equations (36) for $j = 0, 1, \ldots, n-1$

$$
\begin{aligned}
-\left(J_1' + 2J_1\partial_x\right)A_j + \left(\partial_x^2 - J_0\partial_x\right)B_j - \left(\xi' + 2\xi\partial_x\right)\alpha_j - \bar{\xi}\partial_x\beta_j &= 2\partial_x A_{j+1} \\
-\left(\partial_x^2 + J_0\partial_x + J_0'\right)A_j + 2\partial_x B_j - \xi\alpha_j + \bar{\xi}\beta_j &= 0 \\
-\left(2\xi\partial_x + \xi'\right)A_j + \xi B_j - \left(\partial_x^2 - J_1 - J_0\partial_x\right)\beta_j &= -\beta_{j+1} \qquad (43) \\
-\left(\bar{\xi}\partial_x + \bar{\xi}'\right)A_j - \bar{\xi}B_j - \left(\partial_x^2 + J_0\partial_x + J_0' - J_1\right)\alpha_j &= -\alpha_{j+1}
\end{aligned}
$$

and for $j = n$ we get the same dynamical equations as in (38). We note from (43) that A_0 is a constant and $\alpha_0 = \beta_0 = 0$. B_j's will get fixed from the second equation in (43). Choosing $A_0 = -1$, we obtain $B_0 = -\frac{1}{2}J_0(x,t)$ and therefore the zeroth order equation in this case takes the form

$$
\partial_t J_1 = J_1' - \frac{1}{2}J_0'' + \frac{1}{2}J_0 J_0'
$$

$$\begin{aligned}\partial_t J_0 &= 0\\ \partial_t \xi &= \xi' - \frac{1}{2}\xi J_0\\ \partial_t \bar{\xi} &= \bar{\xi}' + \frac{1}{2}\bar{\xi} J_0\end{aligned} \tag{44}$$

Using the recursion relation (43) we determine $A_1 = \frac{1}{2}J_1 - \frac{1}{4}J_0' + \frac{1}{8}J_0^2$, $\alpha_1 = -\bar{\xi}' - \frac{1}{2}\bar{\xi}J_0$, $\beta_1 = -\xi' + \frac{1}{2}\xi J_0$ and $B_1 = \frac{1}{4}J_1' - \frac{1}{8}J_0'' + \frac{1}{4}J_1 J_0 + \frac{1}{16}J_0^3 + \frac{1}{2}\bar{\xi}\xi$. The first order equations can be determined from the dynamical equations (38) to have the form

$$\begin{aligned}\partial_t J_1 &= \frac{1}{4}J_1''' - \frac{3}{2}J_1 J_1' + \frac{3}{2}\xi\bar{\xi}'' - \frac{3}{2}\xi''\bar{\xi} + \frac{3}{4}J_1' J_0' - \frac{3}{8}J_1' J_0^2 + \frac{3}{4}J_1 J_0''\\ &\quad - \frac{3}{4}J_1 J_0 J_0' + \frac{1}{8}J_0 J_0''' - \frac{3}{16}J_0^3 J_0' + \frac{3}{16}J_0^2 J_0'' + \frac{3}{8}J_0 (J_0')^2 - \frac{1}{8}J_0'''' + \frac{3}{2}(J_0\xi\bar{\xi})'\\ \partial_t J_0 &= 0\\ \partial_t \xi &= \xi''' - \frac{3}{4}J_1'\xi - \frac{3}{2}J_1\xi' - \frac{1}{8}J_0''\xi - \frac{3}{2}J_0\xi'' - \frac{3}{4}J_0'\xi' + \frac{3}{4}J_1 J_0 \xi\\ &\quad + \frac{3}{8}J_0^2\xi' + \frac{1}{16}J_0^3\xi\\ \partial_t \bar{\xi} &= \bar{\xi}''' - \frac{3}{4}J_1'\bar{\xi} - \frac{3}{2}J_1\bar{\xi}' + \frac{7}{8}J_0''\bar{\xi} + \frac{3}{2}J_0\bar{\xi}'' + \frac{9}{4}J_0'\bar{\xi}' - \frac{3}{4}J_1 J_0\bar{\xi}\\ &\quad + \frac{3}{8}J_0^2\bar{\xi}' - \frac{1}{16}J_0^3\bar{\xi} + \frac{3}{4}J_0 J_0'\bar{\xi}\end{aligned} \tag{45}$$

Eq.(45) resembles again as one of the $N = 2$ sKdV equation (Eq.(2.27) of ref.[20]) which is not supersymmetric. One can check indeed that they are identical with the redefinitions, $\partial_t \to -\frac{1}{4}\partial_t$, $\partial_x \to \partial_x$, $J_0 \to 2i\phi$, $J_1 \to u + i\phi'$, $\xi \to \frac{1}{\sqrt{2}}(\xi_1 + \xi_2)$, $\bar{\xi} \to \frac{1}{\sqrt{2}}(\xi_1 - \xi_2)$. The first Hamiltonian structure in this case can be read out from (43) to be,

$$\begin{aligned}\{J_1(x),\, J_1(y)\}_1 &= 2\partial_x \delta(x-y)\\ \{\xi(x),\, \bar{\xi}(y)\}_1 &= -\delta(x-y)\end{aligned} \tag{46}$$

and the second Hamiltonian structure remains the same as in (42). So, like in the bosonic case, the zero curvature formulation leads to two bi–Hamiltonian systems.

We have seen that $N = 2$ sKdV–hierarchy can be obtained from the sTB–hierarchy through some redefinition of fields, but this is quite expected since the Hamiltonian structures are related to each other by replacing $J_1 \to J_1 + \frac{1}{2}J_0'$. But, we would like to emphasize that the choice of gauge fixing eq.(34) is not at all obvious as compared to the gauge fixing in $N = 2$ sKdV–hierarchy [20] and further, we had to make a nonlinear field redefinition eq.(11) to recast the hierarchy in the proper form. Our procedure also clarifies how other supersymmetric integrable systems which are known to be contained in sTB–hierarchy

(for example, supersymmetric Non–Linear Schrodinder equation [22]) can be obtained in the zero curvature approach. Finally, in the following we would like to make an observation on the topological symmetry of sTB–hierarchy. It can be readily verified that the sTB–hierarchy equation (36) is invariant under the following two sets of supersymmetry transformations (note that these symmetries respect only one of the hierarchies obtained by $\phi \to \phi + \lambda$),

$$\begin{array}{rclrcl}
\bar{\delta} J_1 &=& 0 & \bar{\delta} A_t^{(-\frac{1}{2})} &=& -\bar{\epsilon} A_t^{(0)} \\
\bar{\delta} \phi &=& -\bar{\epsilon} \bar{\xi} & \bar{\delta} A_t^{-} &=& -\bar{\epsilon} A_t^{-\frac{1}{2}} \\
\bar{\delta} \bar{\xi} &=& 0 & \bar{\delta} A_t^{(0)} &=& 0 \\
\bar{\delta} \xi &=& \bar{\epsilon} J_1 & \bar{\delta} A_t^{-\frac{1}{2}} &=& 0
\end{array} \tag{47}$$

and

$$\begin{array}{rclrcl}
\delta J_1 &=& \epsilon \xi' & \delta A_t^{-} &=& -\epsilon A_t^{(-\frac{1}{2})} \\
\delta \phi &=& \epsilon \xi & \delta A_t^{(0)} &=& -\epsilon \partial_x A_t^{(-\frac{1}{2})} \\
\delta \bar{\xi} &=& \epsilon \left(J_1 - J_0' \right) & \delta A_t^{-\frac{1}{2}} &=& \epsilon \left(A_t^{(0)} - \partial_x A_t^{-} \right) \\
\delta \xi &=& 0 & \delta A_t^{(-\frac{1}{2})} &=& 0
\end{array} \tag{48}$$

We note, first of all, that both these symmetries are of nilpotent type i.e. $\delta^2 = \bar{\delta}^2 = 0$. Also, from the first set of equation in (47), we notice that both J_1 (which can be identified with energy–momentum tensor) and $\bar{\xi}$ (fermionic current) are $\bar{\delta}$-exact, the basic ingredients of a topological field theory [7,8]. In fact, from the Hamiltonian structure (42), we can read out the two symmetry charges of (47) and (48) to be $\int dx\, \bar{\xi}(x)$ and $\int dx\, \xi(x)$, which have conformal weights zero and one respectively much like the topological Q and G-charges of a topological field theory.

To conclude, we have shown how TB–hierarchy can be obtained from the zero curvature condition associated with the gauge group $SL(2,R) \otimes U(1)$. sTB–hierarchy has similarly been shown to follow from the zero curvature condition of the gauge supergroup $OSp(2|2)$. It is shown that both these integrable systems contain two bi–Hamiltonian structures. We have also made the field identifications by which sTB–hierarchy can be converted to $N = 2$ sKdV–hierarchy and thus pointed out that sTB–hierarchy is nothing but the twisted $N = 2$ sKdV–hierarchy. Topological algebras arise naturally as the second Hamiltonain structure in our zero curvature formulation of both TB and sTB–hierarchy indicating a close relationship of these integrable models with 2d topological theories.

ACKNOWLEDGEMENTS:

We would like to thank J. C. Brunelli for discussions as well as for collaboration in the earlier stages of the work. One of us (S.R.) also thanks W. -J. Huang for discussions at an early stage of this work and J. Mas for interesting comments. The work of S.R. is supported in part by the Spanish Ministry of Education (MEC) fellowship. This work was supported in part by the U.S. Department of Energy grant DE-FG02-91ER40685.

REFERENCES:

1. M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71 (1980) 315.

2. L. D. Fadeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, (Springer–Verlag, 1987).

3. A. Das, *Integrable Models*, (World Scientific, Singapore, 1989).

4. L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, (World Scientific, Singapore, 1991).

5. J. L. Gervais and A. Neveu, Nucl. Phys. B209 (1982) 125.

6. D. Gross and A. Migdal, Nucl. Phys. B340 (1990) 333; E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144; M. R. Douglas, Phys. Lett. B238 (1990) 176.

7. E. Witten, Commun. Math. Phys. 117 (1988) 353; Commun. Math. Phys. 118 (1988) 411; T. Eguchi and S. K. Yang, Mod. Phys. Lett. A4 (1990) 1693; R. Dijkgraaf and E. Witten, Nucl. Phys. B342 (1990) 486.

8. R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435; Preprint PUPT-1217/IASSNS-HEP-90/80.

9. E. Witten, Surv. Diff. Geom. 1 (1991) 243.

10. M. Kontsevich, Peprints MPI/91-47, MPI/91-77.

11. B. A. Kupershmidt, Commun. Math. Phys. 99 (1985) 51.

12. L. Bonora and C. S. Xiong, Phys. Lett. B285 (1992) 191; Int. Jour. Mod. Phys. A8 (1993) 2973; H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, Nucl. Phys. B402 (1993) 85.

13. L. J. F. Broer, Appl. Sci. Res. 31 (1975) 377.

14. D. J. Kaup, Progr. Theor. Phys. 54 (1975) 396.

15. J. C. Brunelli and A. Das, Phys. Lett. B337 (1994) 303.

16. J. C. Brunelli and A. Das, Phys. Lett. B354 (1995) 307; Int. J. Mod. Phys. A10 (1995) 4563.

17. S. S. Chern and C. K. Peng, Manus. Math. 28 (1975) 145.

18. S. Okubo and A. Das, Phys. Lett. B209 (1988) 31; A. Das and S. Okubo, Ann. Phys. 190 (1989) 215.

19. A. Das, W. -J. Huang and S. Roy, Int. Jour. Mod. Phys. A7 (1992) 3447.

20. A. Das, W. -J. Huang and S. Roy, Int. Jour. Mod. Phys. A7 (1992) 4293.

21. C. A. Laberge and P. Mathieu, Phys. Lett. B215 (1988) 718.

22. S. Krivonos and A. Sorin, Preprint JINR E2-95-172, hep-th/9504084.